Lossless Coding with Generalized Criteria

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Outline

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5. Conclusions and Future Directions

Outline

Coding problems considered I

Two lossless coding problems are considered, in which the pay-offs are the following:

Problem1: A convex combination of the maximum codeword length and the average codeword length

Find codeword length vector $\ell^{S,\dagger} \in \mathcal{L}(\mathbb{R}^n)$ which solves

$$\mathbb{L}_{\alpha}^{MO}(\mathbf{l}, \mathbf{p}) \triangleq \left\{ \alpha \max_{x \in \mathcal{X}} l(x) + (1-\alpha) \sum_{x \in \mathcal{X}} l(x) p(x) \right\}$$

where $\alpha \in [0,1]$ is a weighting parameter.

The extreme cases:

- $\alpha = 0$ corresponds to the average codeword length, and
- $\alpha = 1$ corresponds to the maximum codeword length.

Coding problems considered II

Problem 2: A convex combination of the average of an exponential function of the codeword length and the average codeword length

Find codeword length vector $\ell^{S,\dagger} \in \mathcal{L}(\mathbb{R}^n)$ which solves

$$\mathbb{L}_{t,\alpha}^{MO}(\mathbf{l},\mathbf{p}) \quad \stackrel{\triangle}{=} \frac{\alpha}{t}\log\Big(\sum_{x\in\mathcal{X}} p(x) D^{t(x)}\Big) + (1-\alpha)\sum_{x\in\mathcal{X}} l(x)p(x)$$

where $\alpha \in [0,1]$ is a weighting parameter and $t \in (-\infty,\infty)$.

Although, Problem 2 is also defined for $t \leq 0$, its solution will be discussed for $t \geq 0$.

Related Problems

The problems considered are multiobjective pay-offs whose solution bridges together an anthology of source coding problems with different pay-offs:

- the average redundancy of the codeword length [6, 2],
- the average of an exponential function of the codeword length [4, 7, 1],
- the average of an exponential function of the redundancy of the codeword length [3, 2, 1].

Related problems

- For $\alpha = 0$ or $\alpha = 1$ the resulting special cases of Problem 2 are found in [6, 4, 7, 3, 2, 1].
- The special case $\mathbb{L}_{t,1}^{MO}(\mathbf{l}$ is the dual problem of universal coding problems formulated as a minimax (see [8]).

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Restructuring Problem 1

Weights

- Define

$$\ell_u = ||\ell||_\infty \triangleq \max_i(\ell_i), \qquad \mathcal{U} \triangleq \{i \in \mathbb{Z}_+^n : \ell_i = \ell_u\}$$

- Pay-off $\mathbb{L}_\alpha^{MO}(\mathbf{l}, \mathbf{p})$

$$\begin{aligned} \alpha\ell_u + (1-\alpha)\sum_i p_i\ell_i &= \left[\alpha + (1-\alpha)\sum_{i\notin\mathcal{U}} p_i\right]\ell_u + \sum_{i\in\mathcal{U}}(1-\alpha)p_i\ell_i \\ &= \sum_i w_i(\alpha)\ell_i, \end{aligned}$$

where $0 \le w_i(\alpha) \le 1,\ \forall\alpha \in [0,1)$ and $\sum_i w_i(\alpha) = 1$

- The code is optimal if $L(\alpha) = \sum_i w_i(\alpha)\ell_i$ is minimal [5].

Optimal weights

Theorem 1

Given the probability vector $p = (p_1 \ p_2 \ \dots \ p_n)$ and $\alpha \in [\alpha_k, \alpha_{k+1})$, $k \in \{0, 1, \dots, n-1\}$, the optimal weights $w^\dagger = (w_1^\dagger \ w_2^\dagger \ \dots \ w_n^\dagger)$ are

$$w_i^\dagger(\alpha) = \begin{cases} (1-\alpha)p_i = w_i(\alpha_k) - (\alpha - \alpha_k)p_i \ , & \forall \ i \notin \mathcal{U}_k \\ w_i^*(\alpha_k) + (\alpha - \alpha_k)\dfrac{\sum_{i \notin \mathcal{U}_k} p_i}{|\mathcal{U}_k|} \ , & \forall \ i \in \mathcal{U}_k \end{cases}$$

where $w_i^*(\alpha) \triangleq \min_{j \in \mathbb{N}_+^n} w_j(\alpha)$,

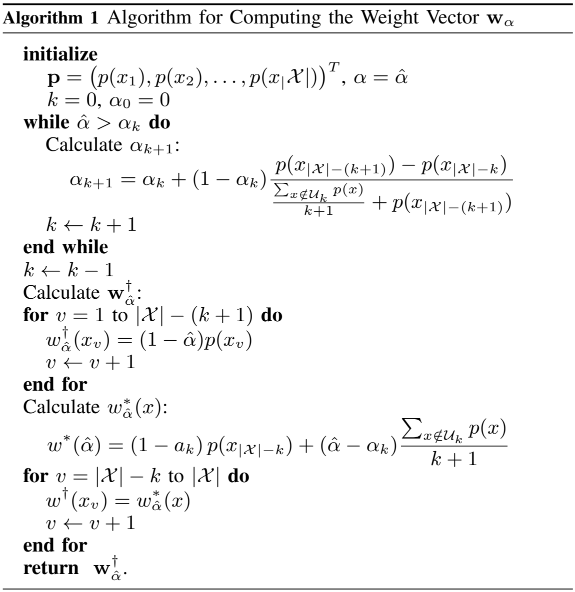
$$\alpha_k \triangleq \min\left\{\alpha \in [0,1) : w_{n-(k-1)}(\alpha) = w_{n-k}(\alpha), \ k \in \{1, \dots, n-1\}\right\}$$

$$\mathcal{U}_k \triangleq \{i \in \{n-k, \dots, n\} : w_i(\alpha_k) = w_i^*(\alpha_k), \ k \in \{0, \dots, n-1\}\}$$

and

$$\alpha_{k+1} = \alpha_k + (1 - \alpha_k)\frac{(p_{n-(k+1)} - p_{n-k})}{\frac{\sum_{i \notin \mathcal{U}_k} p_i}{|\mathcal{U}_k|} + p_{n-(k+1)}}$$

Algorithm for optimal weights



Algorithm 1 Algorithm for Computing the Weight Vector $\mathbf{w}_\alpha$

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initialize
    $\mathbf{p} = \big(p(x_1), p(x_2), \dots, p(x_{|}\mathcal{X}|)\big)^T$, $\alpha = \hat{\alpha}$
    $k = 0$, $\alpha_0 = 0$
while $\hat{\alpha} > \alpha_k$ do
    Calculate $\alpha_{k+1}$:
        $\alpha_{k+1} = \alpha_k + (1 - \alpha_k) \frac{p(x_{|\mathcal{X}|-(k+1)}) - p(x_{|\mathcal{X}|-k})}{\frac{\sum_{x \notin \mathcal{U}_k} p(x)}{k+1} + p(x_{|\mathcal{X}|-(k+1)})}$
    $k \leftarrow k + 1$
end while
$k \leftarrow k - 1$
Calculate $\mathbf{w}^{\dagger}_{\hat{\alpha}}$:
for $v = 1$ to $|\mathcal{X}| - (k+1)$ do
    $w^{\dagger}_{\hat{\alpha}}(x_v) = (1 - \hat{\alpha}) p(x_v)$
    $v \leftarrow v + 1$
end for
Calculate $w^{*}_{\hat{\alpha}}(x)$:
    $w^{*}(\hat{\alpha}) = (1 - a_k)\, p(x_{|\mathcal{X}|-k}) + (\hat{\alpha} - \alpha_k) \frac{\sum_{x \notin \mathcal{U}_k} p(x)}{k+1}$
for $v = |\mathcal{X}| - k$ to $|\mathcal{X}|$ do
    $w^{\dagger}(x_v) = w^{*}_{\hat{\alpha}}(x)$
    $v \leftarrow v + 1$
end for
return $\mathbf{w}^{\dagger}_{\hat{\alpha}}$.
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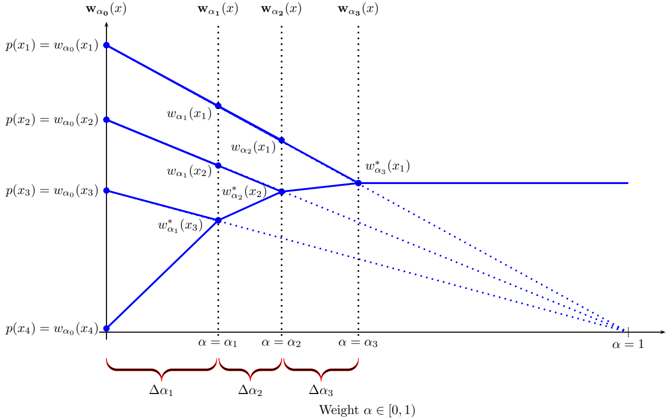
A schematic representation



A schematic representation of the weights for different values of α.

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Optimal Codeword Lengths I

The optimal real-valued codeword length vectors $\mathbf{l} \in \mathcal{L}(\mathbb{R}_+^{|\mathcal{X}|})$ of the multiobjective pay-off of Problem 1 are shown to be

$$l_i(\alpha) = -\log_2 w_i(\alpha).$$

Theorem 2

Consider Problem 1. For any probability distribution $\mathbf{p} \in \mathbb{P}(\mathcal{X})$ and $\alpha \in [0, 1]$ the optimal prefix real-valued code $\mathbf{l} \in \mathbb{R}_+^{|\mathcal{X}|}$ minimizing the pay-off $\mathbb{L}_\alpha^{MO}(\mathbf{l}, \mathbf{p})$ is given by

$$l_\alpha^\dagger(x) = \begin{cases} -\log\Big((1-\alpha)p(x)\Big) & \text{for } x \notin \mathcal{U}_k \\ -\log\left(\frac{\alpha + (1-\alpha)\sum_{x \in \mathcal{U}_k} p(x)}{|\mathcal{U}_k|}\right) & \text{for } x \in \mathcal{U}_k \end{cases}$$

where $\alpha \in [\alpha_k, \alpha_{k+1}) \subset [0, 1], \ \forall k \in \{1, \ldots, |\mathcal{X}| - 1\}$.

Optimal Codeword Lengths II

Theorem 3

Consider Problem 2. For any probability distribution $\mathbf{p} \in \mathbb{P}(\mathcal{X})$ and $\alpha \in [0,1]$ the optimal prefix real-valued code $\mathbf{l} \in \mathbb{R}_+^{|\mathcal{X}|}$ minimizing the pay-off $\mathbb{L}_{t,\alpha}^{MO}(\mathbf{l},\mathbf{p})$ is given by

$$l_{t,\alpha}^{\dagger}(x) = -\log\Big(\alpha \nu_{t,\alpha}(x) + (1-\alpha)p(x)\Big), \quad x \in \mathcal{X} \tag{1}$$

where $\{\nu_{t,\alpha}(x) : x \in \mathcal{X}\}$ is defined via the tilted probability distribution

$$\nu_{t,\alpha}(x) \triangleq \frac{D^{t\, l_{t,\alpha}^{\dagger}(x)} p(x)}{\sum_{x \in \mathcal{X}} p(x) D^{t\, l_{t,\alpha}^{\dagger}(x)}}, \ x \in \mathcal{X} \tag{2}$$

Optimal Codeword Lengths III

Remarks

- *The Limiting Case as $t \to \infty$*:
  The minimization of the multiobjective pay-off $\mathbb{L}_{\alpha}^{MO}(\mathbf{l}, \mathbf{p})$ obtained in Theorem 2 is indeed obtained from the minimization of the two parameter multiobjective pay-off $\mathbb{L}_{t,\alpha}^{MO}(\mathbf{l}, \mathbf{p})$ in the limit, as $t \to \infty$.

$$\lim_{t\to\infty} \mathbb{L}_{t,\alpha}^{MO}(\mathbf{l}, \mathbf{p}) = \mathbb{L}_{\alpha}^{MO}(\mathbf{l}, \mathbf{p}), \ \forall\ \mathbf{l} \text{ and hence at } \mathbf{l} = \mathbf{l}^{\dagger}.$$

- *Coding Theorems*:
  Coding theorems for Problem 1 and Problem 2 can be easily obtained either from the closed form solutions or by following [4].

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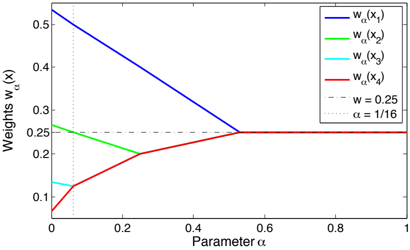
Examples I

Consider binary codewords and a source with $|\mathcal{X}| = 4$ and probability distribution

$$\mathbf{p} = \left(\begin{array}{cccc} \frac{8}{15} & \frac{4}{15} & \frac{2}{15} & \frac{1}{15} \end{array} \right).$$

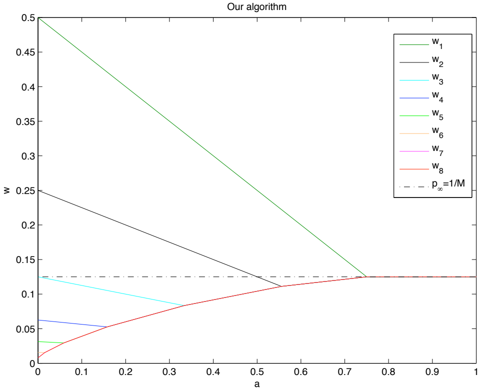


- $\alpha = \alpha_1 = 1/16$:
  Optimal weights are:
  $w_3^\dagger(\alpha) = w_4^\dagger(\alpha) = 1/8$,
  $w_2^\dagger(\alpha) = (1-\alpha)p_2 = 1/4$
  and
  $w_1^\dagger(\alpha) = (1-\alpha)p_1 = 1/2$.
- The resulting codeword lengths correspond to the optimal Huffman code.

Examples II

Consider binary codewords and a source with $|\mathcal{X}| = 8$ and probability distribution

$$\mathbf{p} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \frac{1}{128} & \frac{1}{128} \end{pmatrix}.$$



Outline

Conclusions and Future Directions

Conclusions

- Two lossless coding problems with generalised pay-offs investigated.
- Real-valued codeword length solutions presented.
- Relations to problems discussed in the literature obtained.

Future Directions

Huffman algorithms which solve this problem:

Find codeword length vector $\ell^{H,\dagger} \in \mathcal{L}(\mathbb{Z}_+^n)$ which solves

$$\mathcal{R}_{cc}^{H}(\ell^{H,\dagger}) \triangleq \min_{\ell^{H,\dagger} \in \mathcal{L}(\mathbb{Z}_+^n)} \Big\{ \alpha ||\ell||_\infty + (1-\alpha) \sum_{j \in \mathcal{X}} \ell_j p_j \Big\}$$

are part of ongoing research.

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